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Hodograph method in non-Newtonian MHD transverse fluid flows

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Abstract. Equations for steady plane flows of non-Newtonian electrically conducting fluids of finite and infinite electrical conductivity are recast in the hodograph plane by using the Legendre transform function of the stream-function when the magnetic field is normal to the flow plane. Four examples are worked out to illustrate the developed theory. Solutions and geometries for these examples are determined.

1. Introduction

This paper deals with the application of the hodograph transformation for solving a system of non-linear partial differential equations governing steady plane incompressible flow of an electrically conducting second-grade fluid in the presence of a transverse magnetic field. Equations governing second grade or order [1, 2] fluids are, in general, of third order, as compared to the second-order Navier–Stokes equations and, therefore, application of the hodograph transformation to these flows is a credit to this transformation technique. W.F. Ames [3] has given an excellent survey of this method together with its applications to various other fields. Recently, Siddiqui et al. [4] used the hodograph and Legendre transformations to study electrically non-conducting plane steady non-Newtonian fluid flows. Also, M.K. Swaminathan et al. [5] applied this approach to transverse MHD Newtonian fluid flows. Since electrical conductivity is finite for most liquid metals and it is also finite for other electrically conducting second-grade fluids to which single-fluid models can be applied, our accounting for finite electrical conductivity makes the flow problem realistic and attractive from both a mathematical and a physical point of view. We have also included electrically conducting second-grade fluids of infinite electrical conductivity to make a thorough hodographic study of these fluid flows and to recognize the dawn of superconductivity in science.

The plan of this paper is as follows: in Sections 2 and 3, following the reformulation of the flow equations for transverse plane flows into a convenient form by using M.H. Martin's [6] perceptive idea of reducing the order of the governing equations the flow equations are transformed to the hodograph plane so that the role of the independent variables x, y and the dependent variables u, v (the two components of the velocity vector field) is interchanged. We introduce a Legendre-transform function of the streamfunction and recast all our equations in the hodograph plane in terms of this transformed function in Section 4. The equation that this function must satisfy is then determined and the results are stated in the form of Theorems I and II. Section 5 is devoted to four applications of this approach.

2. Equations of motion

The flow of a homogeneous electrically conducting incompressible second-grade fluid, in the presence of a magnetic field, is governed by

$$\operatorname{div} \mathbf{V} = 0, \tag{1}$$

$$\varrho \dot{\mathbf{V}} = \operatorname{div} \mathbf{T} + \mu^*(\operatorname{curl} \mathbf{H}) \times \mathbf{H}, \tag{2}$$

$$\frac{\partial \mathbf{H}}{\partial t} = \operatorname{curl}(\mathbf{V} \times \mathbf{H}) + \frac{1}{\mu^* \sigma} \nabla^2 \mathbf{H}, \tag{3}$$

$$\operatorname{div} \mathbf{H} = 0 \tag{4}$$

and the constitutive equation for the Cauchy stress $\mathbf{T}$,

$$\mathbf{T} = -p\mathbf{I} + \mu \mathbf{A}_1 + \alpha_1 \mathbf{A}_2 + \alpha_2 \mathbf{A}_1^2.$$

Here $\mathbf{V}$ is the velocity field vector, $\mathbf{H}$ the magnetic vector field, p the dynamic pressure function, ϱ the constant fluid field density, μ the coefficient of dynamic viscosity, μ^* the constant magnetic permeability, σ the electrical conductivity, and α_1, α_2 are the normal stress moduli. The Rivlin–Ericksen tensors $\mathbf{A}_1$ and $\mathbf{A}_2$ are defined as

$$\mathbf{A}_1 = \nabla \mathbf{V} + (\nabla \mathbf{V})^T, \quad \mathbf{A}_2 = \dot{\mathbf{A}}_1 + (\nabla \mathbf{V})^T \mathbf{A}_1 + \mathbf{A}_1 (\nabla \mathbf{V}).$$

Equations (1) to (3) form a system of seven equations in seven unknowns $\mathbf{V}$, $\mathbf{H}$ and p. Equation (4) is an additional condition on $\mathbf{H}$ expressing the absence of magnetic poles in the flow.

Steady plane transverse flow

A steady plane flow in the (x, y) plane is said to be a transverse flow if the magnetic field vector is perpendicular to the (x, y) plane which contains the fluid flow vector field and all the flow variables are functions of x and y. Considering our flow to be steady plane transverse flow, we take $\mathbf{V} = (u(x, y), v(x, y), 0)$, $\mathbf{H} = (0, 0, H(x, y))$ and $\partial/\partial z \equiv 0$.

Introducing the two-dimensional vorticity function $\omega(x, y)$ and a generalized energy function $e(x, y)$ defined by

$$\omega(x, y) = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y},$$

$$e(x, y) = \tfrac{1}{2}\varrho q^2 - \alpha_1(u\nabla^2 u + v\nabla^2 v) - \tfrac{1}{4}(3\alpha_1 + 2\alpha_2)|\mathbf{A}_1|^2 + p + \mu^* \frac{H^2}{2} \tag{5}$$

where $q^2 = u^2 + v^2$, $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ and

$$|\mathbf{A}_1|^2 = 4\left(\frac{\partial u}{\partial x}\right)^2 + 4\left(\frac{\partial v}{\partial y}\right)^2 + 2\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2,$$

and introducing the definiton of **V** and **H** into the above system of equations, we obtain the system

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad \text{(continuity)}$$

$$\frac{\partial e}{\partial x} = \varrho v\omega - \mu \frac{\partial \omega}{\partial y} - \alpha_1 v \nabla^2 \omega,$$

$$\qquad \text{(linear momentum)}$$

$$\frac{\partial e}{\partial y} = -\varrho u\omega + \mu \frac{\partial \omega}{\partial x} - \alpha_1 u \nabla^2 \omega, \tag{6}$$

$$u \frac{\partial H}{\partial x} + v \frac{\partial H}{\partial y} - \nu_H \left(\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} \right) = 0, \quad \text{(diffusion)}$$

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \omega, \qquad \text{(vorticity)}$$

where $\nu_H = (\mu^* \sigma)^{-1}$, of five partial differential equations in five unknown functions u, v, ω, H and e as functions of x, y. Once a solution for these is found, the pressure function is determined from the expression for $e(x, y)$ given in (5). This system of equations governs steady plane transverse flows of an incompressible second-grade fluid of finite electrical conductivity. For the motion of a second-grade fluid of infinite electrical conductivity, we only replace the diffusion equation in the above system of equations by $u(\partial H/\partial x) + v(\partial H/\partial y) = 0$ since $\nu_H \to 0$ for such fluid flows.

3. Equations in the hodograph plane

Letting the flow variables $u(x, y)$, $v(x, y)$ be such that, in the region of flow, the Jacobian

$$J(x, y) = \frac{\partial(u, v)}{\partial(x, y)} \neq 0, \quad 0 < |J| < \infty, \tag{7}$$

we may consider x and y as functions of u and v. By means of $x = x(u, v)$, $y = y(u, v)$, we derive the following relations:

$$\frac{\partial u}{\partial x} = J \frac{\partial y}{\partial v}, \ \frac{\partial u}{\partial y} = -J \frac{\partial x}{\partial v}, \ \frac{\partial v}{\partial x} = -J \frac{\partial y}{\partial u}, \ \frac{\partial v}{\partial y} = J \frac{\partial x}{\partial u}. \tag{8}$$

We also obtain the relations

$$\frac{\partial g}{\partial x} = \frac{\partial(g, y)}{\partial(x, y)} = \mathbf{J} \frac{\partial(\mathbf{g}, y)}{\partial(u, v)}, \ \frac{\partial g}{\partial y} = -\frac{\partial(g, x)}{\partial(u, v)} = \mathbf{J} \frac{\partial(x, \mathbf{g})}{\partial(u, v)}, \tag{9}$$

where $g = g(x, y) = g(x(u, v), y(u, v)) = \mathbf{g}(u, v)$ is any continuously differentiable function and

$$J = J(x, y) = \frac{\partial(u, v)}{\partial(x, y)} = \left[\frac{\partial(x, y)}{\partial(u, v)}\right]^{-1} = \mathbf{J}(u, v) \tag{10}$$

Using these transformation relations for the first-order partial derivatives appearing in system (6) and the transformation equations for the functions $\boldsymbol{\omega}$, $\mathbf{H}$, $\mathbf{e}$, defined by

$$\omega(x, y) = \omega(x(u, v), y(u, v)) = \boldsymbol{\omega}(u, v),$$

$$H(x, y) = H(x(u, v), y(u, v)) = \mathbf{H}(u, v),$$

$$e(x, y) = e(x(u, v), y(u, v)) = \mathbf{e}(u, v),$$

the system (6) is transformed into the following system in the (u, v) plane:

$$\frac{\partial x}{\partial u} + \frac{\partial y}{\partial v} = 0, \tag{11}$$

$$\mathbf{J}\frac{\partial(\mathbf{e}, y)}{\partial(u, v)} = \varrho v\boldsymbol{\omega} - \mu\mathbf{J}w_1 - \alpha_1 v\mathbf{J}\left[\frac{\partial(x, \mathbf{J}w_1)}{\partial(u, v)} + \frac{\partial(\mathbf{J}w_2, y)}{\partial(u, v)}\right], \tag{12}$$

$$\mathbf{J}\frac{\partial(x, \mathbf{e})}{\partial(u, v)} = -\varrho u\boldsymbol{\omega} + \mu\mathbf{J}w_2 + \alpha_1 u\mathbf{J}\left[\frac{\partial(x, \mathbf{J}w_1)}{\partial(u, v)} + \frac{\partial(\mathbf{J}w_2, y)}{\partial(u, v)}\right], \tag{13}$$

$$uG_1 + vG_2 - \nu_H\left[\frac{\partial(\mathbf{J}G_1, y)}{\partial(u, v)} + \frac{\partial(x, \mathbf{J}G_2)}{\partial(u, v)}\right] = 0, \tag{14}$$

$$\mathbf{J}(x_v - y_u) = \boldsymbol{\omega}, \tag{15}$$

where

$$\begin{aligned} w_1 &= w_1(u, v) = \frac{\partial(x, \boldsymbol{\omega})}{\partial(u, v)}, \quad w_2 = w_2(u, v) = \frac{\partial(\boldsymbol{\omega}, y)}{\partial(u, v)}, \\ G_1 &= G_1(u, v) = \frac{\partial(\mathbf{H}, y)}{\partial(u, v)}, \quad G_2 = G_2(u, v) = \frac{\partial(x, \mathbf{H})}{\partial(u, v)}. \end{aligned} \tag{16}$$

System of equations (11) to (15) is a system of five equations for the five unknown functions x, y, $\boldsymbol{\omega}$, $\mathbf{H}$, $\mathbf{e}$ of u, v, when $\mathbf{J}$, w_1, w_2, G_1 and G_2 are eliminated using (10) and (16). Once a solution $x = x(u, v)$, $y = y(u, v)$, $\boldsymbol{\omega} = \boldsymbol{\omega}(u, v)$, $\mathbf{H} = \mathbf{H}(u, v)$, $\mathbf{e} = \mathbf{e}(u, v)$ is obtained, we are lead to the solutions $u = u(x, y)$, $v = v(x, y)$ and therefore $\boldsymbol{\omega} = \boldsymbol{\omega}(u(x, y), v(x, y)) = \omega(x, y)$, $\mathbf{H} = \mathbf{H}(u(x, y), v(x, y)) = H(x, y)$, $\mathbf{e} = \mathbf{e}(u(x, y), v(x, y)) = e(x, y)$ for the system of equations (6) governing the finitely conducting flow. The above analysis also holds true for infinitely conducting second-grade fluid flows. However, for these flows, the diffusion equation is replaced by $uG_1 + vG_2 = 0$.

4. Equations for the Legendre transform function and H(*u*, *v*)

The equation of continuity implies the existence of a streamfunction $\psi(x, y)$ such that

$$d\psi = -v dx + u dy \quad \text{or} \quad \frac{\partial \psi}{\partial x} = -v, \quad \frac{\partial \psi}{\partial y} = u. \tag{17}$$

Likewise, equation (11) implies the existence of a function $L(u, v)$, called the Legendre transform function of the streamfunction $\psi(x, y)$, so that

$$dL = -y du + x dv \quad \text{or} \quad \frac{\partial L}{\partial u} = -y, \quad \frac{\partial L}{\partial v} = x, \tag{18}$$

and the two functions $\psi(x, y)$, $L(u, v)$ are related by

$$L(u, v) = vx - uy + \psi(x, y). \tag{19}$$

Introducing $L(u, v)$ into the system (11)–(15), with $\mathbf{J}$, w_1, w_2, G_1, G_2 given by (10), (16) respectively, it follows that (11) is identically satisfied and the system may be replaced by

$$\mathbf{J}\frac{\partial\left(\frac{\partial L}{\partial u}, \mathbf{e}\right)}{\partial(u, v)} = \varrho v\omega - \mu \mathbf{J} w_1 - \alpha_1 v \mathbf{J}\left[\frac{\partial\left(\frac{\partial L}{\partial v}, \mathbf{J}w_1\right)}{\partial(u, v)} + \frac{\partial\left(\frac{\partial L}{\partial u}, \mathbf{J}w_2\right)}{\partial(u, v)}\right], \tag{20}$$

$$\mathbf{J}\frac{\partial\left(\frac{\partial L}{\partial v}, \mathbf{e}\right)}{\partial(u, v)} = -\varrho u\omega - \mu \mathbf{J} w_2 + \alpha_2 u \mathbf{J}\left[\frac{\partial\left(\frac{\partial L}{\partial v}, \mathbf{J}w_1\right)}{\partial(u, v)} + \frac{\partial\left(\frac{\partial L}{\partial u}, \mathbf{J}w_2\right)}{\partial(u, v)}\right], \tag{21}$$

$$uG_1 + vG_2 - \nu_H\left[\frac{\partial\left(\frac{\partial L}{\partial u}, \mathbf{J}G_1\right)}{\partial(u, v)} - \frac{\partial\left(\frac{\partial L}{\partial v}, \mathbf{J}G_2\right)}{\partial(u, v)}\right] = 0, \tag{22}$$

$$\mathbf{J}\left[\frac{\partial^2 L}{\partial v^2} + \frac{\partial^2 L}{\partial u^2}\right] = \omega, \tag{23}$$

where now

$$\mathbf{J} = \left[\frac{\partial^2 L}{\partial v^2}\frac{\partial^2 L}{\partial u^2} - \left(\frac{\partial^2 L}{\partial u \partial v}\right)^2\right]^{-1},$$

$$w_1 = \frac{\partial\left(\dfrac{\partial L}{\partial v}, \omega\right)}{\partial(u, v)}, \quad w_2 = \frac{\partial\left(\dfrac{\partial L}{\partial u}, \omega\right)}{\partial(u, v)}, \tag{24}$$

$$G_1 = \frac{\partial\left(\dfrac{\partial L}{\partial u}, \mathbf{H}\right)}{\partial(u, v)}, \quad G_2 = \frac{\partial\left(\dfrac{\partial L}{\partial v}, \mathbf{H}\right)}{\partial(u, v)},$$

for four unknown functions $L(u, v)$, $\omega(u, v)$, $\mathbf{H}(u, v)$ and $\mathbf{e}(u, v)$ after eliminating $\mathbf{J}$, w_1, w_2, G_1 and G_2.

By using the integrability condition

$$\left[\mathbf{J}\frac{\partial^2 L}{\partial u \partial v}\frac{\partial}{\partial v} - \mathbf{J}\frac{\partial^2 L}{\partial v^2}\frac{\partial}{\partial u}\right]\left(\mathbf{J}\frac{\partial\left(\dfrac{\partial L}{\partial u}, \mathbf{e}\right)}{(u, v)}\right) = \left[\mathbf{J}\frac{\partial^2 L}{\partial u^2}\frac{\partial}{\partial v} - \mathbf{J}\frac{\partial^2 L}{\partial u \partial v}\frac{\partial}{\partial u}\right]\left(\mathbf{J}\frac{\partial\left(\dfrac{\partial L}{\partial v}, \mathbf{e}\right)}{\partial(u, v)}\right),$$

i.e., $\partial^2 e/\partial x \partial y = \partial^2 e/\partial y \partial x$ in the (x, y)-plane, we eliminate $\mathbf{e}(u, v)$ from (20), (21) and obtain

$$\mu\left[\frac{\partial\left(\dfrac{\partial L}{\partial v}, \mathbf{J}w_1\right)}{\partial(u, v)} + \frac{\partial\left(\dfrac{\partial L}{\partial u}, \mathbf{J}w_2\right)}{\partial(u, v)}\right]$$

$$+ \alpha_1\left[v\frac{\partial\left(\dfrac{\partial L}{\partial v}, \mathbf{J}\left\{\partial\left(\dfrac{\partial L}{\partial v}, \mathbf{J}w_1\right)\Big/\partial(u, v) + \partial\left(\dfrac{\partial L}{\partial u}, \mathbf{J}w_2\right)\Big/\partial(u, v)\right\}\right)}{\partial(u, v)}\right.$$

$$\left.+ u\frac{\partial\left(\dfrac{\partial L}{\partial u}, \mathbf{J}\left\{\partial\left(\dfrac{\partial L}{\partial v}, \mathbf{J}w_1\right)\Big/\partial(u, v) + \partial\left(\dfrac{\partial L}{\partial u}, \mathbf{J}w_2\right)\Big/\partial(u, v)\right\}\right)}{\partial(u, v)}\right] = \varrho(uw_2 + vw_1) \tag{25}$$

where $\mathbf{J}$, w_1, w_2 are given in (24). Summing up, we have the following theorem:

THEOREM I. *If $L(u, v)$ is the Legendre transform function of a stream-function of steady, plane, transverse, incompressible, finitely conducting second-grade fluid flows and* $\mathbf{H}(u, v)$ *is the transformed magnetic field vector component function, then $L(u, v)$ and* $\mathbf{H}(u, v)$ *must satisfy equations (25) and (22) where $\omega(u, v)$,* $\mathbf{J}(u, v)$*, $w_1(u, v)$, $w_2(u, v)$, $G_1(u, v)$, $G_2(u, v)$ are given by (23), (24).*

If the fluid has infinite electrical conductivity, then the transformed diffusion equation becomes

$$uG_1 + vG_2 = 0 \tag{26}$$

where G_1, G_2 are given in (24). Hence, we have the following theorem.

THEOREM II. *If $L(u, v)$ is the Legendre transform function of a stream-function of steady, plane, transverse, incompressible, infinitely conducting fluid flows and $\mathbf{H}(u, v)$ is the transformed magnetic field vector component function, then $L(u, v)$, $\mathbf{H}(u, v)$ must satisfy equations (25) and (26), where $\omega(u, v)$, $\mathbf{J}(u, v)$, $w_1(u, v)$, $w_2(u, v)$, $G_1(u, v)$, $G_2(u, v)$ are given by (23), (24).*

Once a solution $L(u, v)$, $\mathbf{H}(u, v)$ is found, for which $\mathbf{J}$ evaluated from (24) satisfies $0 < |\mathbf{J}| < \infty$, the solutions for the velocity components are obtained by solving equation (18) simultaneously. Once the velocity components $u = u(x, y)$, $v = v(x, y)$ are obtained, we have $H(x, y)$ in the physical plane from the solution for $\mathbf{H}(u, v)$ in the hodograph plane. We then determine the vorticity and the energy function by using $\mathbf{V}(x, y)$ in the definition of vorticity and the linear momentum equations in system (6), respectively. Finally, the pressure function is obtained from the expression for $e(x, y)$.

We now develop the flow equations in polar coordinates in the hodograph plane. Defining

$$u + iv = q\mathrm{e}^{i\theta} \tag{27}$$

we get the following transformations:

$$\begin{gathered}\frac{\partial}{\partial u} = \cos\theta\frac{\partial}{\partial q} - \frac{\sin\theta}{q}\frac{\partial}{\partial\theta}, \quad \frac{\partial}{\partial v} = \sin\theta\frac{\partial}{\partial q} + \frac{\cos\theta}{q}\frac{\partial}{\partial\theta}, \\ \frac{\partial(F, G)}{\partial(u, v)} = \frac{\partial(F^*, G^*)}{\partial(q, \theta)}\frac{\partial(q, \theta)}{\partial(u, v)} = \frac{1}{q}\frac{\partial(F^*, G^*)}{\partial(q, \theta)}\end{gathered} \tag{28}$$

where $F(u, v) = F^*(q, \theta)$, $G(u, v) = G^*(q, \theta)$ are continuously differentiable functions. On using these relations, and regarding (q, θ) as new independent variables, the expressions for $\mathbf{J}$, ω, w_1, w_2, G_1, G_2 in the (q, θ) plane take the form

$$J^*(q, \theta) = q^4\left[q^2\frac{\partial^2 L^*}{\partial q^2}\left(q\frac{\partial L^*}{\partial q} + \frac{\partial^2 L^*}{\partial\theta^2}\right) - \left(\frac{\partial L^*}{\partial\theta} - q\frac{\partial^2 L^*}{\partial q\partial\theta}\right)^2\right]^{-1},$$

$$\omega^*(q, \theta) = J^*\left[\frac{\partial^2 L^*}{\partial q^2} + \frac{1}{q^2}\frac{\partial^2 L^*}{\partial\theta^2} + \frac{1}{q}\frac{\partial L^*}{\partial q}\right],$$

$$w_1^*(q, \theta) = \frac{1}{q}\frac{\partial\left(\sin\theta\dfrac{\partial L^*}{\partial q} + \dfrac{\cos\theta}{q}\dfrac{\partial L^*}{\partial q}, \omega^*\right)}{\partial(q, \theta)},$$

$$w_2^*(q, \theta) = \frac{1}{q}\frac{\partial\left(\cos\theta\dfrac{\partial L^*}{\partial q} - \dfrac{\sin\theta}{q}\dfrac{\partial L^*}{\partial\theta}, \omega^*\right)}{\partial(q, \theta)},$$

$$G_1^*(q, \theta) = \frac{1}{q}\frac{\partial\left(\cos\theta\dfrac{\partial L^*}{\partial q} - \dfrac{\sin\theta}{q}\dfrac{\partial L^*}{\partial\theta}, H^*\right)}{\partial(q, \theta)},$$

$$G_2^*(q, \theta) = \frac{1}{q}\frac{\partial\left(\sin\theta\dfrac{\partial L^*}{\partial q} + \dfrac{\cos\theta}{q}\dfrac{\partial L^*}{\partial\theta}, H^*\right)}{\partial(q, \theta)}, \tag{29}$$

Equations (25) and (22) are transformed to the (q, θ) plane as

$$\mu\chi^* + \alpha_1\left[\sin\theta\frac{\partial\left(\sin\theta\dfrac{\partial L^*}{\partial q} + \dfrac{\cos\theta}{q}\dfrac{\partial L^*}{\partial\theta}, J^*\chi^*\right)}{\partial(q, \theta)}\right.$$

$$\left. + \cos\theta\frac{\partial\left(\cos\theta\dfrac{\partial L^*}{\partial q} - \dfrac{\sin\theta}{q}\dfrac{\partial L^*}{\partial\theta}, J^*\chi^*\right)}{\partial(q, \theta)}\right] = pq(\cos\theta w_2^* + \sin\theta w_1^*), \tag{30}$$

$$q(\cos\theta G_1^* + \sin\theta G_2^*) - \frac{\nu_H}{q}\left[\frac{\partial\left(\cos\theta\dfrac{\partial L^*}{\partial q} - \dfrac{\sin\theta}{q}\dfrac{\partial L^*}{\partial\theta}, J^*G_1^*\right)}{\partial(q, \theta)}\right.$$

$$\left. + \frac{\partial\left(\sin\theta\dfrac{\partial L^*}{\partial q} + \dfrac{\cos\theta}{q}\dfrac{\partial L^*}{\partial q}, J^*G_2^*\right)}{\partial(q, \theta)}\right] = 0, \tag{31}$$

where χ^* is defined as

$$\chi^*(q, \theta) = \frac{1}{q}\left[\frac{\partial\left(\sin\theta\dfrac{\partial L^*}{\partial q} + \dfrac{\cos\theta}{q}\dfrac{\partial L^*}{\partial\theta}, J^*w_1^*\right)}{\partial(q, \theta)}\right.$$

$$\left. + \frac{\partial\left(\cos\theta\dfrac{\partial L^*}{\partial q} - \dfrac{\sin\theta}{q}\dfrac{\partial L^*}{\partial\theta}, J^*w_2^*\right)}{\partial(q, \theta)}\right], \tag{32}$$

Having developed the above transformations, we state the following corollaries which respectively follow from Theorem I and II.

COROLLARY I. *If $L^*(q, \theta)$ and $H^*(q, \theta)$ are the Legendre transform function of a streamfunction and the magnetic field vector component function, respectively, of the equations governing* *the motion of steady plane transverse flows of incompressible finitely conducting second-grade fluids, then* $L^*(q, \theta)$ *and* $H^*(q, \theta)$ *must satisfy equations (30) and (31) where* J^*, ω^*, w_1^*, w_2^*, G_1^*, G_2^*, χ^* *are given by (29) and (32).*

COROLLARY II. *If* $L^*(q, \theta)$ *and* $H^*(q, \theta)$ *are the Legendre transform function of a stream-function and the magnetic field vector component function, respectively, of the equations governing the motion of steady plane transverse flows of incompressible infinitely conducting second-grade fluids, then* $L^*(q, \theta)$ *and* $H^*(q, \theta)$ *must satisfy equations (30) and*

$$\cos\theta G_1^* + \sin\theta G_2^* = 0 \tag{33}$$

where J^*, ω^*, w_1^*, w_2^*, G_1^*, G_2^*, χ^* *are given by (29) and (32).*

Once a solution $L^*(q, \theta)$, $H^*(q, \theta)$ is known, we employ the relations

$$x = \sin\theta \frac{\partial L^*}{\partial q} + \frac{\cos\theta}{q}\frac{\partial L^*}{\partial \theta}, \quad y = \frac{\sin\theta}{q}\frac{\partial L^*}{\partial \theta} - \cos\theta \frac{\partial L^*}{\partial q} \tag{34}$$

and (27) to obtain the velocity components $u = u(x, y)$, $v = v(x, y)$ in the physical plane. Having obtained the velocity components, we get $H(x, y)$ in the x, y plane from $H^*(q, \theta)$. The other flow variables are then determined by using the flow equations in the physical plane.

5. Applications

In this section, we consider some of the flow problems as applications of Theorems I and II and Corollaries I and II.

Application I

Let

$$L(u, v) = Au^2 + Bv^2 \tag{35}$$

be the Legendre transform function, where A, B are arbitrary constants and A, B are non-zero. Using (35) in equations (23), (24), we obtain

$$\mathbf{J} = \frac{1}{4AB}, \quad \omega = \frac{A+B}{2AB}, \quad w_1 = w_2 = 0, \quad G_1 = 2A\frac{\partial \mathbf{H}}{\partial v}, \quad G_2 = -2B\frac{\partial \mathbf{H}}{\partial u}. \tag{36}$$

We now consider the finitely conducting and the infinitely conducting case separately by applying Theorem I and II, respectively.

Finitely conducting fluid

Employing (35), (36) in equations (25) and (22), we find that (25) is identically satisfied and $\mathbf{H}(u, v)$ must satisfy

$$\frac{u}{B}\frac{\partial \mathbf{H}}{\partial v} - \frac{v}{A}\frac{\partial \mathbf{H}}{\partial u} - \nu_H\left[\frac{1}{2B^2}\frac{\partial^2 \mathbf{H}}{\partial v^2} + \frac{1}{2A^2}\frac{\partial^2 \mathbf{H}}{\partial u^2}\right] = 0. \tag{37}$$

Assuming $\mathbf{H}(u, v) = F(u) + G(v)$ to be the form of a possible solution for $\mathbf{H}(u, v)$, we find that (37) becomes

$$\frac{u}{B}G'(v) - \frac{v}{A}F'(u) - \nu_H\left[\frac{1}{2B^2}G''(v) + \frac{1}{2A^2}F''(u)\right] = 0. \tag{38}$$

Differentiating twice with respect to u, we get

$$\frac{F'''(u)}{A}v + \frac{\nu_H}{2A^2}F^{(iv)}(u) = 0.$$

The above equation holds true for all v if

$$\frac{1}{A}F'''(u) = 0, \quad \frac{\nu_H}{2A^2}F^{(iv)}(u) = 0. \tag{39}$$

Therefore, we have

$$F(u) = \frac{C_1}{2}u^2 + C_2 u + C_3,$$

where C_1, C_2, C_3 are arbitrary constants.

Using $F(u)$ so obtained in equation (38), we find that $G(v)$ must satisfy

$$\left\{\frac{G'}{B} - \frac{C_1 v}{A}\right\}u - \left\{\frac{C_2}{A}v + \frac{\nu_H}{2B^2}G'' + \frac{\nu_H C_1}{2A^2}\right\} = 0.$$

This equation holds true for all u if

$$\frac{G'}{B} - \frac{C_1 v}{A} = 0, \quad \frac{C_2}{A}v + \frac{\nu_H}{2B^2}G'' + \frac{\nu_H C_1}{2A^2} = 0. \tag{40}$$

Solving equations (40), we obtain

$$G(v) = -C_1 v^2 + C_4, \quad C_2 = 0 \quad \text{and} \quad A = -B.$$

Therefore, we have

$$L(u, v) = A(u^2 - v^2), \quad \mathbf{H}(u, v) = \frac{C_1}{2}(u^2 - v^2) + C_5$$

where $C_5 = C_3 + C_4$.

Using $L(u, v) = A(u^2 - v^2)$ in (18) and solving the resulting equations simultaneously, we get

$$\mathbf{V} = (u, v) = \left(-\frac{y}{2A}, -\frac{x}{2A}\right). \tag{41}$$

Employing (41) in the solution for $\mathbf{H}(u, v)$, we obtain

$$H(x, y) = \frac{C_1}{8A^2}(y^2 - x^2) + C_5. \tag{42}$$

Using $\omega = 0$, equation (41) in the linear momentum equations in system (6) and integrating, we obtain $e(x, y)$. Employing this solution for $e(x, y)$ and (41) in (5), the pressure function is determined to be

$$p(x, y) = C_6 - \frac{\varrho}{8A^2}(x^2 + y^2) + \frac{3\alpha_1 + 2\alpha_2}{2A^2} - \frac{\mu^*}{2}\left[\frac{C_1}{8A^2}(y^2 - x^2) + C_5\right]^2 \tag{43}$$

where C_6 is constant.

Infinitely conducting fluid

Employing (35), (36) in equations (25), (26), equation (25) is identically satisfied and (26) takes the form

$$\frac{u}{B}\frac{\partial \mathbf{H}}{\partial v} - \frac{v}{A}\frac{\partial \mathbf{H}}{\partial u} = 0. \tag{44}$$

A solution for $\mathbf{H}(u, v)$ is

$$\mathbf{H}(u, v) = \varphi\left(\frac{Bv^2 + Au^2}{2}\right) \tag{45}$$

where φ is an arbitrary function of its argument.

Using $L(u, v) = Au^2 + Bv^2$ in (18) and solving simultaneously for u, v, we get

$$\mathbf{V} = (u, v) = \left(-\frac{y}{2A}, \frac{x}{2B}\right). \tag{46}$$

Following the same procedure as in finitely conducting fluid flow, we obtain

$$H(x, y) = \varphi\left(\frac{x^2}{8B} + \frac{y^2}{8A}\right) \tag{47}$$

and

$$p(x, y) = \frac{\varrho}{8AB}\left[\frac{y^2}{A} + \frac{x^2}{B}\right] + \frac{3\alpha_1 + 2\alpha_2}{8}\left(\frac{A - B}{AB}\right)^2 - \frac{\mu^*}{2}\left[\varphi\left(\frac{x^2}{8B} + \frac{y^2}{8A}\right)\right]^2 + C_7 \tag{48}$$

where C_7 is an arbitrary constant.

Summing up, we have the following theorems:

THEOREM III. *If $L(u, v) = A(u^2 - v^2)$ is the Legendre transform function of a streamfunction for a steady, plane, transverse, incompressible, finitely conducting second-grade fluid flow, then the flow in the physical plane is a flow with hyperbolic streamlines with flow variables given by (41) to (43).*

THEOREM IV. *If $L(u, v) = Au^2 + Bv^2$ is the Legendre transform function of a streamfunction for a steady, plane, transverse, incompressible, infinitely conducting second-grade fluid flow, then the flow in the physical plane is a flow with flow variables given by (46) to (48) having*

$$\frac{x^2}{4B} + \frac{y^2}{4A} = \textit{constant}$$

as its streamlines.

Application II

We let

$$L(u, v) = Auv + Bu^2 + Cu + D \tag{49}$$

to be the Legendre transform function, where A, B, C, D are arbitrary constants and $A \neq 0$.

Evaluating $\mathbf{J}$, ω, w_1, w_2, G_1 and G_2, by using (49) in equations (23), (24), we have

$$\mathbf{J} = -\frac{1}{A^2}, \quad \omega = -\frac{2B}{A^2}, \quad w_1 = w_2 = 0, \quad G_1 = 2B\frac{\partial \mathbf{H}}{\partial v} - v\frac{\partial \mathbf{H}}{\partial u}, \quad G_2 = A\frac{\partial \mathbf{H}}{\partial v}. \tag{50}$$

Finitely conducting fluid

Using (49), (50) in equations (25) and (22), we find that (25) is identically satisfied and $\mathbf{H}(u, v)$ must satisfy

$$[2Bu + Av]\frac{\partial \mathbf{H}}{\partial v} - Au\frac{\partial \mathbf{H}}{\partial u} + \nu_H\left[\frac{\partial^2 \mathbf{H}}{\partial u^2} + \left(1 + \frac{4B^2}{A^2}\right)\frac{\partial^2 H}{\partial v^2} - \frac{4B}{A}\frac{\partial^2 H}{\partial u \partial v}\right] = 0. \quad (51)$$

A solution for $\mathbf{H}(u, v)$ satisfying (51) is obtained to be

$$\mathbf{H}(u, v) = D_1 \int \exp\left[\frac{A}{2\nu_H}u^2\right]\mathrm{d}u + D_2$$

where D_1, D_2 are arbitrary constants.

Proceeding as in Application I, we get

$$\mathbf{V} = (u, v) = \left(\frac{x}{A}, -\left(\frac{2B}{A^2}x + \frac{y}{A} + \frac{C}{A}\right)\right),$$

$$H(x, y) = \frac{D_1}{A}\int \exp\left[\frac{1}{2A\nu_H}x^2\right]\mathrm{d}x + D_2, \quad (52)$$

$$p(x, y) = -\frac{\varrho}{2A^2}(x^2 + y^2) - \frac{\varrho C}{A^2}y + (6\alpha_1 + 4\alpha_2)\frac{A^2 + B^2}{A^4}$$

$$- \frac{\mu^*}{2}\left[\frac{D_1}{A}\int \exp\left[\frac{x^2}{2A\nu_H}\right]\mathrm{d}x + D_2\right]^2 + D_3$$

where D_3 is an arbitrary constant.

Infinitely conducting fluid

In this case, only the diffusion equation is replaced by

$$(2Bu + Av)\frac{\partial \mathbf{H}}{\partial v} - Au\frac{\partial \mathbf{H}}{\partial u} = 0.$$

Solving this equation, we obtain

$$\mathbf{H}(u, v) = \phi(Bu^2 + Auv)$$

where ϕ is an arbitrary function of its argument.

We proceed as before and obtain

$$\mathbf{V} = (u, v) = \left(\frac{x}{A}, -\left(\frac{2B}{A^2}x + \frac{y}{A} + \frac{C}{A}\right)\right),$$

$$H(x, y) = \phi\left(xy + \frac{B}{A}x^2 + Cx\right)$$

and

$$p(x, y) = -\frac{\varrho}{2A^2}(x^2 + y^2) - \frac{\varrho C}{A^2}y + (6\alpha_1 + 4\alpha_2)\frac{A^2 + B^2}{A^4}$$
$$- \frac{\mu^*}{2}\left[\phi\left(xy + \frac{B}{A}x^2 + Cx\right)\right]^2 + D_4 \tag{53}$$

where D_4 is an arbitrary constant.

Summing up, we have the following theorem:

THEOREM V. *If $L(u, v) = Auv + Bu^2 + Cu + D$ is the Legendre transform function of a streamfunction for a steady, plane, transverse, incompressible, finitely conducting second-grade fluid flow, then the flow in the physical is given by equation (52) with $(x/A)(y + Bx/A + C) =$ constant as its streamlines.*

THEOREM VI. *If $L(u, v) = Auv + Bu^2 + Cu + D$ is the Legendre transform function of a streamfunction for a steady, plane, transverse, incompressible, infinitely conducting second-grade fluid flow, then the flow in the physical plane is given by equations (53) with $(x/A)(y + Bx/A + C) =$ constant as its streamlines.*

Application III

Let

$$L^*(q, \theta) = F(q), \quad F'(q) \neq 0, \quad F''(q) \neq 0. \tag{54}$$

Employing (54) in equations (29) to evaluate J^*, ω^*, w_1^*, w_2^*, G_1^*, G_2^*, x and y, we obtain

$$J^* = \frac{q}{F'(q)F''(q)}, \quad \omega^* = \frac{qF''(q) + F'(q)}{F'(q)F''(q)},$$

$$w_1^* = -\frac{1}{q}\omega^{*\prime}\cos\theta F'(q), \quad w_2^* = \frac{1}{q}\omega^{*\prime}\sin\theta F'(q),$$

$$G_1^* = \frac{1}{q}\left[\cos\theta F''(q)\frac{\partial H^*}{\partial\theta} + \sin\theta F'(q)\frac{\partial H^*}{\partial q}\right], \tag{55}$$

$$G_2^* = \frac{1}{q}\left[\sin\theta F''(q)\frac{\partial H^*}{\partial\theta} - \cos\theta F'(q)\frac{\partial H^*}{\partial q}\right],$$

$$x = F'(q)\sin\theta, \quad y = -F'(q)\cos\theta.$$

We now study finitely conducting fluid flow and infinitely conducting fluid flow as applications of Corollaries I and II.

Finitely conducting fluid

Employing (54), (55) in equations (30) and (31), we find that $F(q)$ and $H^*(q, \theta)$ must satisfy

$$\omega^{*\prime} + F'(q)\left[\frac{\omega^{*\prime}}{F''(q)}\right]' = 0, \tag{56}$$

$$F''_{(q)}\frac{\partial H^*}{\partial \theta} - \frac{v_H}{q}[\cos\theta F''_{(q)}(J^*G_1^*)_\theta + \sin\theta F'(q)(J^*G_1^*)_q$$
$$+ \sin\theta F''_{(q)}(J^*G_2^*)_\theta - \cos\theta F'(q)(J^*G_2^*)_q] = 0, \tag{57}$$

so that $F(q)$ is the Legendre transform function of a streamfunction.

Since equation (56) is identically satisfied when $\omega^{*\prime} = 0$ and can be rewritten as

$$\frac{\omega^{*\prime\prime}}{\omega^{*\prime}} + \frac{F''(q)}{F'(q)} - \frac{F'''(q)}{F''(q)} = 0 \tag{58}$$

when $\omega^{*\prime} \neq 0$, it follows that we have to deal separately with $L^*(q, \theta) = F(q)$ having variable vorticity and $L^*(q, \theta) = F(q)$ having constant vorticity.

Case 1. (Variable vorticity). By the expressions for x and y in (55), we get

$$r = \sqrt{x^2 + y^2} = \pm F'(q), \quad \frac{dr}{dq} = \pm F''(q). \tag{59}$$

We integrate (58) twice with respect to q, and obtain

$$\omega^* = E_1 \ln|F'(q)| + E_2 \tag{60}$$

where E_1 and E_2 are arbitrary constants.

Substituting for ω^* given in (55) into (60), we have

$$q + r\frac{dq}{dr} = \pm[E_1 r \ln r + E_2 r], \tag{61}$$

since $F''(q) \neq 0$ and, therefore, $dr/dq = (dq/dr)^{-1}$.

Integrating (61), we get

$$q = \pm\left[\frac{E_1}{2} r \ln r + \left(\frac{2E_2 - E_1}{4}\right) r + \frac{E_3}{r}\right] \tag{62}$$

where E_3 is an arbitrary constant.

Using (59), (62) in equation (34) and making use of the definitions $u = q\cos\theta$, $v = q\sin\theta$ and $\omega = \partial v/\partial x - \partial u/\partial y$, we get

$$\begin{aligned} u(x,y) &= -y\left[\frac{E_1}{4}\ln(x^2+y^2) + \left(\frac{2E_2-E_1}{4}\right) + \frac{E_3}{x^2+y^2}\right], \\ v(x,y) &= x\left[\frac{E_1}{4}\ln(x^2+y^2) + \left(\frac{2E_2-E_1}{4}\right) + \frac{E_3}{x^2+y^2}\right], \\ \omega(x,y) &= \frac{E_1}{2}\ln(x^2+y^2) + E_2. \end{aligned} \tag{63}$$

Transforming the diffusion equation (57) back to the (x, y)-plane, we find that $H(x, y)$ must satisfy

$$\left[\frac{E_1}{4}\ln(x^2+y^2) + \left(\frac{2E_2-E_1}{4}\right) + \frac{E_3}{x^2+y^2}\right]\left(x\frac{\partial H}{\partial y} - y\frac{\partial H}{\partial x}\right) - \nu_H\left(\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2}\right) = 0. \tag{64}$$

A solution for $H(x, y)$ is found to be

$$H(x,y) = E_4\ln\left(\frac{x^2+y^2}{2}\right) + E_5 \tag{65}$$

where E_4, E_5 are arbitrary constants.

Using (63) in the linear momentum equations of system (6) and integrating, we get $e(x, y)$. Using this solution for $e(x, y)$ and (65) in (5), the pressure function is determined to be

$$\begin{aligned} p(x,y) = {} & \frac{\varrho}{8}E_1\left[E_3 + \frac{E_1}{4}(x^2+y^2)\right][\ln(x^2+y^2)]^2 \\ & + \frac{\varrho}{2}\left[E_2E_3 - \frac{E_1E_3}{2} + \left(\frac{E_1E_2 - E_1^2}{4}\right)(x^2+y^2)\right]\ln(x^2+y^2) \\ & + \frac{\varrho}{4}\left[\frac{E_2^2}{2} + \frac{5E_1^2}{8} - E_1E_2\right](x^2+y^2) - \frac{\varrho E_3}{2}(x^2+y^2)^{-1} \\ & - \mu E_1\tan^{-1}\left(\frac{x}{y}\right) + \alpha_1E_1\left[\frac{E_1}{4}\ln(x^2+y^2) + \left(\frac{2E_2-E_1}{4}\right) + \frac{E_3}{x^2+y^2}\right] \\ & + \frac{3\alpha_1+2\alpha_2}{2}\left[\frac{E_1^2}{4} + \frac{4E_3^2}{(x^2+y^2)^2} - \frac{E_1E_3}{x^2+y^2}\right] \\ & - \frac{\mu^*}{2}\left[E_4\ln\left(\frac{x^2+y^2}{2}\right) + E_5\right]^2 + E_6 \end{aligned} \tag{66}$$

where E_6 is arbitrary constant.

Case 2. (Constant vorticity). Using $\omega^* = \omega_0 =$ constant in the expression for ω^* given in (55), we get

$$qF''(q) + F'(q) - \omega_0 F'(q)F''(q) = 0. \tag{67}$$

We integrate equation (67) with respect to q and obtain

$$\omega_0 F'^2(q) - 2qF'(q) + 2E_7 = 0 \tag{68}$$

where E_7 is an arbitrary constant.

Employing $F'(q)$ given by (59) in equation (68) and solving for q, we have

$$q = \pm\left[\frac{\omega_0}{2}\sqrt{x^2 + y^2} + \frac{E_7}{\sqrt{x^2 + y^2}}\right]. \tag{69}$$

Proceeding as in the variable vorticity case, we obtain

$$u(x, y) = -y\left[\frac{\omega_0}{2} + \frac{E_7}{x^2 + y^2}\right], \quad v(x, y) = x\left[\frac{\omega_0}{2} + \frac{E_7}{x^2 + y^2}\right], \tag{70}$$

$$H(x, y) = E_4 \ln\left(\frac{x^2 + y^2}{2}\right) + E_5, \tag{71}$$

$$p(x, y) = \varrho\left[\frac{\omega_0^2}{8}(x^2 + y^2) + \frac{\omega_0 E_7}{2}\ln(x^2 + y^2) - \frac{E_7^2}{2(x^2 + y^2)} - \frac{\omega_0 E_7}{2}\right] + \frac{(6\alpha_1 + 4\alpha_2)E_7^2}{(x^2 + y^2)^2} - \frac{\mu^*}{2}\left[E_4 \ln\left(\frac{x^2 + y^2}{2}\right) + E_5\right]^2 + E_8 \tag{72}$$

where E_8 is an arbitrary constant.

Infinitely conducting fluid

In this case, the transformed diffusion equation reduces to

$$\frac{\partial H^*}{\partial \theta} = 0.$$

Therefore, we have

$$H(q, \theta) = \phi(q)$$

where ϕ is an arbitrary function of its argument. In the (x, y)-plane, we have

$$H(x, y) = \phi(q) \tag{73}$$

where q is given by (62) for $L^*(q, \theta) = F(q)$ having variable vorticity and q is given by (69) for $L^*(q, \theta) = F(q)$ having constant vorticity.

Therefore, we obtain the flow variables as follows:

(i) Variable vorticity
$u(x, y)$, $v(x, y)$, $\omega(x, y)$, $H(x, y)$ are given by (63), (73) and

$$p(x, y) = \frac{\varrho}{8} E_1 \left[E_3 + \frac{E_1}{4}(x^2 + y^2) \right] [\ln (x^2 + y^2)]^2 + \frac{\varrho}{2} \left[E_2 E_3 - \frac{E_1 E_3}{2} + \left(\frac{E_1 E_2 - E_1^2}{4} \right)(x^2 + y^2) \right] \ln (x^2 + y^2) + \frac{\varrho}{4} \left[\frac{E_2^2}{2} + \frac{5E_1^2}{8} - E_1 E_2 \right] (x^2 + y^2) - \frac{\varrho E_3}{2} (x^2 + y^2)^{-1} - \mu E_1 \tan^{-1}\left(\frac{x}{y}\right) + \alpha_1 E_1 \left[\frac{E_1}{4} \ln (x^2 + y^2) + \left(\frac{2E_2 - E_1}{4} \right) + \frac{E_3}{x^2 + y^2} \right] + \frac{3\alpha_1 + 2\alpha_2}{2} \left[\frac{E_1^2}{4} + \frac{4E_3^2}{(x^2 + y^2)^2} - \frac{E_1 E_3}{x^2 + y^2} \right] - \frac{\mu^*}{2} [\phi(q)]^2 + E_9 \qquad (74)$$

where q is given by (62) and E_9 is an arbitrary constant.

(ii) Constant vorticity
$u(x, y)$, $v(x, y)$, $\omega(x, y)$, $H(x, y)$ are given by (70) and (73) and

$$p(x, y) = \varrho \left[\frac{\omega_0^2}{8}(x^2 + y^2) + \frac{\omega_0 E_7}{2} \ln (x^2 + y^2) - \frac{E_7^2}{2(x^2 + y^2)} - \frac{\omega_0 E_7}{2} \right] + \frac{(6\alpha_1 + 4\alpha_2)E_7^2}{(x^2 + y^2)^2} - \frac{\mu^*}{2} [\phi(q)]^2 + E_{10} \qquad (75)$$

where q is given by (69) and E_{10} is an arbitrary constant. Summing up, we have the following theorems:

THEOREM VII. *If $L^*(q, \theta) = F(q)$ is the Legendre transform function of a streamfunction for a steady, plane, transverse, incompressible, finitely conducting second-grade fluid flow, then the flow in the physical plane is*

(a) given by equations (63), (65) and (66) with

$$(x^2 + y^2) \left[\frac{E_1}{8} \ln (x^2 + y^2) + \frac{E_2 - E_1}{4} \right] + \frac{E_3}{2} \ln (x^2 + y^2) = \text{constant}$$

as its streamlines, when vorticity is not a constant;

(b) given by equations (70) to (72) having

$$\omega_0 (x^2 + y^2) + 2E_7 \ln (x^2 + y^2) = \text{constant}$$

as its streamlines, when vorticity is a constant.

THEOREM VIII. *If $L^*(q, \theta) = F(q)$ is the Legendre transform function of a streamfunction for a steady, plane, transverse, incompressible, infinitely conducting second-grade fluid flow, then the flow in the physical plane is*

(*a*) *given by equations (63), (73), (74) with*

$$(x^2 + y^2)\left[\frac{E_1}{8}\ln(x^2 + y^2) + \frac{E_2 - E_1}{4}\right] + \frac{E_3}{2}\ln(x^2 + y^2) = constant$$

as its streamlines, when vorticity is not a constant;

(*b*) *given by equations (70), (73), (75) with*

$$\omega_0(x^2 + y^2) + 2E_7 \ln(x^2 + y^2) = constant$$

as its streamlines, when vorticity is a constant.

Application IV

Let

$$L^*(q, \theta) = A\theta + B \tag{76}$$

be the Legendre transform function, where A, B are arbitrary constants and $A \neq 0$.

Using (76) in equations (29), we get

$$J^* = -\frac{q^4}{A^2}, \quad \omega^* = w_1^* = w_2^* = 0,$$

$$G_1^* = \frac{1}{q}\left[\frac{A}{q^2}\sin\theta\frac{\partial H^*}{\partial\theta} + \frac{A}{q}\cos\theta\frac{\partial H^*}{\partial q}\right], \tag{77}$$

$$G_2^* = \frac{1}{q}\left[\frac{A}{q}\sin\theta\frac{\partial H^*}{\partial q} - \frac{A}{q^2}\cos\theta\frac{\partial H^*}{\partial\theta}\right].$$

Finitely conducting fluid

Using (76), (77) in equations (30) and (31), we find that (30) is identically satisfied and $H^*(q, \theta)$ must satisfy

$$A\frac{\partial H^*}{\partial q} - \nu_H\left[\frac{A}{q^2}\sin\theta(J^*G_1^*)_\theta + \frac{A}{q}\cos\theta(J^*G_1^*)_q\right.$$

$$\left. + \frac{A}{q}\sin\theta(J^*G_2^*)_q - \frac{A}{q^2}\cos\theta(J^*G_2^*)_\theta\right] = 0. \tag{78}$$

Using $L^*(q, \theta) = A\theta + B$ in equations (34) and making use of (27), we obtain

$$u(x, y) = \frac{Ax}{x^2 + y^2}, \quad v(x, y) = \frac{Ay}{x^2 + y^2}. \tag{79}$$

Transforming (78) back to (x, y)-plane, we have that $H(x, y)$ must satisfy

$$\frac{Ax}{x^2 + y^2}\frac{\partial H}{\partial x} + \frac{Ay}{x^2 + y^2}\frac{\partial H}{\partial y} - \nu_H\left(\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2}\right) = 0. \tag{80}$$

Writing (80) in polar coordinates in the (x, y)-plane, we have $H(r, \theta)$ satisfying

$$\frac{\partial^2 H}{\partial r^2} + \left(1 + \frac{A}{\nu_H}\right)\frac{1}{r}\frac{\partial H}{\partial r} + \frac{1}{r^2}\frac{\partial^2 H}{\partial \theta^2} = 0. \tag{81}$$

A solution for $H(r, \theta)$ satisfying (81) is

$$H(r, \theta) = M_1\left[\frac{\theta^2}{2} + \frac{\nu_H}{A}\ln r\right] + M_2\theta + M_3\frac{\nu_H}{A}r^{A/\nu_H} + M_4 \tag{82}$$

where M_1, M_2, M_3, M_4 are arbitrary constants.

Employing $\omega = 0$, (79) in the linear momentum equations and integrating we obtain $e(x, y)$. Therefore, the pressure function is determined from the expression for $e(x, y)$ to be

$$\begin{aligned} p(r, \theta) = {} & M_5 - \frac{\varrho A^2}{2r^2} + (3\alpha_1 + 2\alpha_2)\frac{2A^2}{r^4} \\ & - \frac{\mu^*}{2}\left[M_1\left(\frac{\theta^2}{2} + \frac{\nu_H}{A}\ln r\right) + M_2\theta + M_3\frac{\nu_H}{A}r^{A/\nu_H} + M_4\right] \end{aligned} \tag{83}$$

where M_5 is an arbitrary constant.

Infinitely conducting fluid

In this case, the transformed diffusion equation is replaced by

$$A\frac{\partial H^*}{\partial q} = 0.$$

Therefore, we get

$$H^*(q, \theta) = \phi(\theta)$$

where ϕ is an arbitrary function of its argument.

In the (x, y)-plane, we have

$$H(x, y) = \phi\left(\tan^{-1}\left(\frac{x}{y}\right)\right). \tag{84}$$

Using $L^*(q, \theta) = A\theta + B$ and proceeding as before, we have

$$\mathbf{V} = (u, v) = \left(\frac{Ax}{x^2 + y^2}, \frac{Ay}{x^2 + y^2}\right) \tag{85}$$

and

$$p(x, y) = M_6 - \frac{\varrho A^2}{2(x^2 + y^2)} + (3\alpha_1 + 2\alpha_2)\frac{2A^2}{(x^2 + y^2)^2} - \frac{\mu^*}{2}\left[\phi\left(\tan^{-1}\left(\frac{y}{x}\right)\right)\right]^2 \tag{86}$$

where M_6 is an arbitrary constant.

Summing up, we have the following theorems:

THEOREM IX. *If $L^*(q, \theta) = A\theta + B$ is the Legendre transform function of a streamfunction for a steady, plane, transverse, incompressible, finitely conducting second-grade fluid flow, then the flow in the physical plane is given by equations (79), (82), (83) having* $\tan^{-1}(x/y) =$ *constant as its streamlines.*

THEOREM X. *If $L^*(q, \theta) = A\theta + B$ is the Legendre transform function of a streamfunction for a steady, plane, transverse, incompressible, infinitely conducting second-grade fluid flow, then the flow in the physical plane is given by equations (84) to (86) with* $\tan^{-1}(x/y) =$ *constant as its streamlines.*

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